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Power law jumps and power law waiting times, fractional calculus and human mobility in epidemiological systems

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Abstract

In human mobility not only power law jumps but also non-exponential waiting times of power law type have been reported to be important, at least in the analysis of surrogate data of such human mobility. More recently much improved algorithms have been developed for power law jump distributions and power law waiting time distributions. We improve the analysis of these methods by avoiding histograms via using less data or simulation hungry ordering methods, similar to what is used e.g. in Kolmogorov-Smirnov tests. Then we investigate these new possibilities to analyse such systems of power laws in jumps and waiting times and its connections with fractional calculus and their potential in analyzing human mobility in application to epidemiology, especially dengue fever epidemiology in Thailand. It turns out that inhomogeneities in population densities already can be used to model human mobility and subsequently epidemiological spreading, e.g. via models mimiking radiation. Such models also present already

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power laws in their jump distributions, and could be combined with information on waiting times to improve accuracy in describing the dengue fever spreading between provinces in Thailand.

Key words: random walks, survival probability, Riesz fractional derivative, Caputo derivative, dengue fever, spatially extended stochastic processes, superdiffusion

1 Introduction

Power law jumps have been suggested to be important in human mobility, especially in the context of epidemiological spreading. Not only power law jumps but also non-exponential waiting times of power law type have been reported to be important, at least in the analysis of surrogate data of such human mobility [1]. The spatio-temporal distribution of e.g. US-Dollar bills shows signs of power law jumps and power law waiting times, and can serve as a surrogat to the underlying human mobility which causes the money bills movement.

More recently much improved algorithms have been developed for power law jump distributions and power law waiting time distributions [2], making it computationally easier now to investigate theoretically such systems with power law random walks, and also help in analyzing empirical data. In the analysis of empirical data often many Monte-Carlos runs of the theory to be tested have to be generated in order to compare the distributions with the empirical data [3]. Hence fast algorithms for drawing random numbers are here of special importance. We improve the analysis of these methods by avoiding histograms via using less data or simulation hungry ordering methods, similar to what is used e.g. in Kolmogorov-Smirnov tests [4].

We investigate these new possibilities to analyse such systems of power laws in jumps and waiting times and its connections with fractional calculus [5, 6] and their potential in analyzing human mobility in application to epidemiology, especially dengue fever epidemiology in Thailand. It turns out that inhomogeneities in population densities already can be used to model human mobility and subsequently epidemiological spreading, e.g. via models mimiking radiation [7]. Such models also present already power laws in their jump distributions, and could be combined with information on waiting times to improve accuracy in describing the dengue fever spreading between provinces in Thailand. Again not the human mobility directly might be accessible, but surrogates as mobile phone connections or twitter data emission could give hints on change of locations and waiting times between moves. Power law jumps can easily be included in spatial models of epidemiological systems in the adjacency matrix of infected being neighbours of susceptibles e.g., see [8] for the set up of such spatial stochastic systems.

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2 Random walks with power law jumps and power law waiting times and fractional calculus

We will first describe random walks in a stochastic frame work and then clear its connection with fractional diffusion processes, before solving the dynamics via Laplace-Fourier transforms.

2.1 Random walks as stochastic processes

Let us assume a general random walk with waiting times τ_i drawn from a distribution $\psi(\tau)$ and jumps ξ_i from a distribution $\lambda(\xi)$. Then the random walker is after n waiting times and n jumps at the position x_n at time t_n with

$$x_n = \sum_{i=1}^{n} \xi_i \qquad , \qquad t_n = \sum_{i=1}^{n} \tau_i \quad . \tag{1}$$

The probability $p(x,t)$ to be at time t at location x is not easily related to the jump position after n steps $p(x_n,t_n)$. However, the arrival probability $\eta(x,t)$ to have arrived at time t at the location x can be related to all previous arrival probabilities $\eta(x',t')$ via waiting between time t' and t, with probability $\psi(t-t')$, and then a jump from x' to x, with probability $\lambda(x-x')$, hence

$$\eta(x,t) = \int_{-\infty}^{\infty} \lambda(x-x') \int_0^t \psi(t-t')\eta(x',t')\, dt'\, dx' + \delta(x)\delta(t) \tag{2}$$

with the last term $\delta(x)\delta(t)$ of having arrived at $x_0 = 0$ at time $t_0 = 0$ and never moved further until time t. Then the probability $p(x,t)$ to be at time t at location x is given by the probability to have arrived before, at time t' at location x, with arrival probability $\eta(x,t')$, and then the survival probability $\Psi(t-t')$ of not having jumped since, between time t' and t, i.e.

$$\Psi(t-t') := 1 - \int_{t'}^{t} \psi(t'')\, dt'' \tag{3}$$

hence

$$p(x,t) = \int_0^t \Psi(t-t')\eta(x,t')\, dt' \tag{4}$$

see e.g. [6]. But now we observe that the dynamic equations for the probabilities, Eq. (2) and Eq. (4), are nothing but convolution integrals. And due to the integration boundaries, in the spatial coordinate x a Fourier convolution and in the temporal coordinate t a Laplace convolution have to be applied.

Explicitly, for the Laplace transform of Eq. (3) we have

$$\bar{\eta}(x,s) = \int_{-\infty}^{\infty} \lambda(x-x')\bar{\psi}(s)\bar{\eta}(x',s)\, dx' + \delta(x)\cdot 1 \tag{5}$$

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and for Eq. (2) under Laplace transform

$$\bar{p}(x,s) = \bar{\Psi}(s) \cdot \bar{\eta}(x,s) \tag{6}$$

and inserting this into Eq. (5) gives

$$\bar{p}(x,s) = \bar{\Psi}(s) \cdot \left(\bar{\psi}(s) \int_{-\infty}^{\infty} \lambda(x-x')\bar{\eta}(x',s)\, dx' + \delta(x) \right) \quad . \tag{7}$$

The convolution under Fourier transform then gives the Laplace-Fourier transform

$$\tilde{\bar{p}}(k,s) = \bar{\Psi}(s) \cdot \left(\bar{\psi}(s)\tilde{\lambda}(k)\tilde{\bar{\eta}}(k,s) + 1 \right) \tag{8}$$

and with $\tilde{\bar{p}}(k,s) = \bar{\Psi}(s) \cdot \tilde{\bar{\eta}}(k,s)$, hence $\tilde{\bar{\eta}}(k,s) = \tilde{\bar{p}}(k,s)/\bar{\Psi}(s)$, hence

$$\tilde{\bar{p}}(k,s) = \bar{\Psi}(s) \cdot \left(\bar{\psi}(s)\tilde{\lambda}(k)\frac{\tilde{\bar{p}}(k,s)}{\bar{\Psi}(s)} + 1 \right) \tag{9}$$

and resolved for $\tilde{\bar{p}}(k,s)$ gives

$$\tilde{\bar{p}}(k,s) = \bar{\Psi}(s) \cdot \frac{1}{1 - \bar{\psi}(s) \cdot \tilde{\lambda}(k)} \tag{10}$$

and with the definition of the survival function, Eq. (3), under Laplace transform finally we obtain

$$\tilde{\bar{p}}(k,s) = \frac{1 - \bar{\psi}(s)}{s} \cdot \frac{1}{1 - \bar{\psi}(s) \cdot \tilde{\lambda}(k)} \tag{11}$$

as a closed solution of the probability in Laplace-Fourier transform. The solution in space and time, $p(x,t)$, will be given below via the respective back transformations, ones we have specified the probabilities $\lambda(\xi)$ and $\psi(\tau)$.

2.2 Power law jump and waiting time distributions

Now taking power law distributions for $\psi(\tau)$ and $\lambda(\xi)$, hence in Laplace transform, respectively in Fourier transform

$$\begin{aligned} \bar{\psi}(s) &= 1 - s^{\nu} \qquad \text{for} \quad s \to 0 \\ \tilde{\lambda}(k) &= 1 - |k|^{\mu} \qquad \text{for} \quad |k| \to 0 \quad , \end{aligned} \tag{12}$$

see [6], we obtain for the Laplace-Fourier transform of the probability

$$\tilde{\bar{p}}(x,s) = \frac{s^{\nu-1}}{s^{\nu} + |k|^{\mu}} \tag{13}$$

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neglecting higher order terms like $s^\nu \cdot |k|^\mu$.

This procedure extracts the asymptotic behaviour of the random walk, e.g. for any elementary jump distribution $\lambda(\xi)$ with a power law dominant moment after summing up large numbers n of them, i.e. $x_n = \sum_{i=1}^n \xi_i$, in the same way as the central limit theorem leads to Gaussian distributions for the sum of any elementary distribution with limited, i.e. finite, second moment. Hence in modelling of large scale behaviour we can start immediately with Gaussian random numbers, saving the extra effort of large scale analysis. Likewise here we can start in modelling of jump processes of power law behaviour for large scales (Lévy stable processes) directly with the power law distributions as elementary steps, as will be done in the next section via drawing power law random numbers.

2.3 Connection with fractional calculus

This Laplace-Fourier transform, Eq. (13), is the same as we obtain for a fractional diffusion equation

$$\frac{\partial^\nu}{\partial t^\nu} u(x,t) = \chi \frac{\partial^\mu}{\partial x^\mu} u(x,t) \tag{14}$$

using the definition of the Riesz fractional derivative via the Fourier transform

$$\frac{\partial^\mu}{\partial x^\mu} e^{ikx} := -|k|^\mu \cdot e^{ikx} \quad . \tag{15}$$

for the spatial part with exponent μ between 2 for ordinary diffusion and smaller else, and the Caputo fractional derivative via the Laplace transform

$$\left(\int_0^\infty e^{-st} \left(\frac{\partial^\nu}{\partial t^\nu} f(t) \right) \, dt \right) := s^\nu \bar{f}(s) - s^{\nu-1} f(0) \tag{16}$$

for the temporal part with ν between 1 for ordinary diffusion and smaller else. For the Laplace-Fourier transform we have then

$$s^\nu \tilde{\bar{u}}(k,s) - s^{\nu-1} \tilde{u}(k, t_0 = 0) = -\chi |k|^\mu \tilde{\bar{u}}(k,s) \tag{17}$$

and hence

$$\tilde{\bar{u}}(k,s) = \tilde{u}(k, t_0 = 0) \frac{s^{\nu-1}}{s^\nu + \chi |k|^\mu} \quad . \tag{18}$$

2.4 Solution

To solve Eq. (18) respectively Eq. (13) we first invert the Laplace transform. From the Laplace transform theory we have

$$E_\nu(ct^\nu) \leftrightarrow \frac{s^{\nu-1}}{s^\nu - c} \quad .$$

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for $c \in C$ and $0 < \nu \leq 2$, where E_ν is the Mittag-Leffler function of order ν, defined in the complex plane by the power series

$$E_\nu(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\nu n + 1)}, \quad \nu > 0, \; z \in C \quad .$$

For $\nu = 1$ the power series is identical to the one for the exponential function. Then we obtain

$$\tilde{u}(k,t) = \tilde{u}(k,t_0) E_\nu \left(-\chi |k|^\mu (t-t_0)^\nu\right))$$

and using Fourier back transform we can find the solution of Eq. (18)

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{u}(k,t_0) E_\nu \left(-\chi |k|^\mu (t-t_0)^\nu\right) \mathrm{e}^{ikx} \, dk \quad , \tag{19}$$

where the initial condition $u(y,t_0)$ also has to be considered in Fourier space

$$\tilde{u}(k,t_0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(y,t_0) \mathrm{e}^{-iky} \, dy \quad . \tag{20}$$

Therefore, we can write the general solution $u(x,t)$ given from the initial distribution $u(y,t_0)$ as

$$u(x,t) = \int_{-\infty}^{\infty} u(y,t_0) G(x-y,t-t_0) \, dy \quad , \tag{21}$$

with the Green's function

$$G(x-y,t-t_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{ik(x-y)} E_\nu \left(-\chi |k|^\mu (t-t_0)^\nu\right) \, dk \quad . \tag{22}$$

Substituting $z := \frac{x-y}{\chi^{\frac{1}{\mu}} (t-t_0)^{\frac{\nu}{\mu}}}$ and $\tilde{k} := k \chi^{\frac{1}{\mu}} (t-t_0)^{\frac{\nu}{\mu}}$ we get

$$\begin{aligned} G(x-y,t-t_0) &= \frac{1}{\chi^{\frac{1}{\mu}} (t-t_0)^{\frac{\nu}{\mu}}} \, \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{i\tilde{k}z} E_\nu(-|\tilde{k}|^\mu) d\tilde{k} \\ &= \frac{1}{\chi^{\frac{1}{\mu}} (t-t_0)^{\frac{\nu}{\mu}}} \, K_{\mu,\nu} \left(\frac{x-y}{\chi^{\frac{1}{\mu}} (t-t_0)^{\frac{\nu}{\mu}}} \right) \end{aligned} \tag{23}$$

where the function

$$K_{\mu,\nu}(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{i\tilde{k}z} E_\nu(-|\tilde{k}|^\mu) d\tilde{k} \tag{24}$$

is a generalization of the Lévy stable function, in which the usual exponential function is generalized to the Mittag-Leffler function. If the initial condition is a Dirac delta function

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then $\tilde{u}(k,0) = 1$ and the solution of Eq. (18) is given by $\tilde{u}(k,t) = E_\nu(-\chi|k|^\mu t^\nu\ )$ for $k \in R$ and $t \geq 0$. Finally we can conclude that the solution of Eq. (18) for $\chi = 1$ is of the form

$$u(x,t) = \frac{1}{t^{\nu/\mu}} K_{\mu,\nu}\left(\frac{x}{t^{\nu/\mu}}\right)$$

$$K_{\mu,\nu}(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{ikz} E_\nu(-|k|^\mu) dk \quad . \tag{25}$$

hence a scaling function of $x/t^{\nu/\mu}$ for the solution $u(x,t)\cdot t^{\nu/\mu}$. We will now investigate the numerical aspects of such power law random walks in more detail.

3 Drawing random numbers from power law distributions

Recently, efficient procedures have been proposed to draw random numbers from Lévy-stable distributions as well as for waiting times with power law tails in as similar way as for the exponential waiting time distribution or the Box-Muller algorithm for Gaussian random numbers in the context of space-time fractional derivatives [2]. Here we demonstrate their use and improve their analysis in terms of computational speed by using the ordering of the drawn random numbers to obtain the cumulative distribution function directly, similar to what is done in the context of the Kolmogorov-Smirnov test [4], instead of binning into small intervals which is computationally very time consuming to obtain good counting numbers in each interval, especially in double logarithmic scale to analyse the power law asymptotics.

Like e.g. for exponential waiting times we can obtain random variables with desired distribution from uniformly distributed random numbers u on the unit interval. For the exponential distribution we have $\tau = -\gamma_t\, ln(u)$ with rate γ_t, and analogously for power law distributed waiting times the Mittag-Leffler random numbers with exponent ν

$$\tau = -\gamma_t\, ln(u)\left(\frac{sin(\nu\pi)}{tan(\nu\pi v)} - cos(\nu\pi)\right)^{\frac{1}{\nu}} \tag{26}$$

with uniformly distributed random numbers u and v on the unit interval. And for the jumps we have for general Lévy stable random numbers with exponent μ an analog to the Box-Muller algorithm for Gaussian random numbers, hence with $\phi := \pi\left(v - \frac{1}{2}\right)$ we have

$$\xi = \gamma_x \left(\frac{-ln(u)\cdot cos(\phi)}{cos((1-\mu)\phi)}\right)^{1-\frac{1}{\mu}} \frac{sin(\nu\phi)}{cos(\phi)} \tag{27}$$

again with uniformly distributed random numbers u and v on the unit interval. In the following we will investigate these random numbers further [2] especially for random walks with small rates $\gamma_t \to 0$ and fixing the diffusion constant to unit via $\gamma_x = \gamma_t^{\nu/\mu}$.

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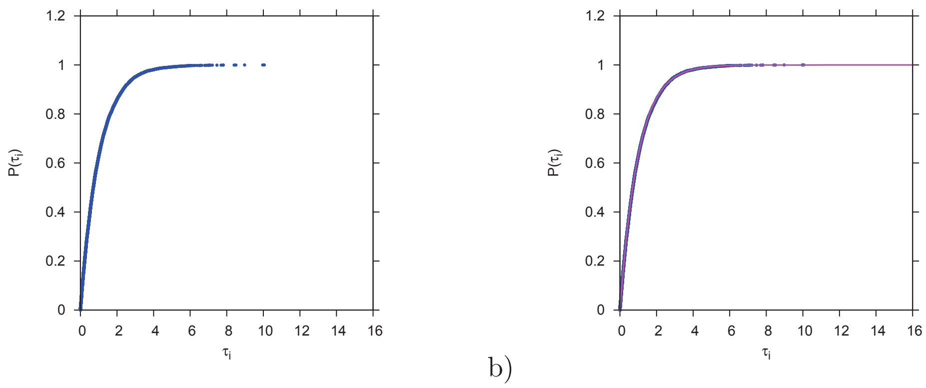

Figure 1: Random numbers drawn from the exponential distribution, hence $\nu = 1$, in a) ordered data in blue, and in b) ordered data in comparison with the theoretical result, curve in pink.

3.1 Drawing random numbers from power law distributions: Waiting time distribution

We now test the asymptotic power laws for large arguments, first for the case of the waiting time distribution $\psi(\tau)$. For exponential waiting times we have

$$\psi(\tau) = a \cdot e^{-a \cdot \tau} \tag{28}$$

and hence the cumulative distribution function

$$P(\tau) := \int_0^\tau \psi(\tilde{\tau})\, d\tilde{\tau} = 1 - e^{-a \cdot \tau} \tag{29}$$

which can be obtained from N data points by ordering the data τ_i on the x-axis and ploting on the y-axis the increasing index i/N, see Fig. 1 a). The resulting curve can be well compared with the theoretical curve $P(\tau) = 1 - e^{-a \cdot \tau}$, see Fig. 1 b), here for the rate $a = 1$.

Next we change the exponent away from $\nu = 1$ for the exponential waiting time to $\nu = 0.9$, see Fig. 2 a), demonstrating the deviation between the observed distribution function from the data in blue and the continuous curve with the theoretical cumulative distribution function of the exponential waiting time. Now the survivial function Ψ is defined as the probability to have waiting times longer than τ, i.e.

$$\Psi(\tau) := \int_\tau^\infty \psi(\tilde{\tau})\, d\tilde{\tau} = 1 - \int_0^\tau \psi(\tilde{\tau})\, d\tilde{\tau} = 1 - P(\tau) \tag{30}$$

and we can plot this in double logarithmic scale, see Fig 2 b), from which the power law tail becomes now visible as straight line for large arguments $\tau \to \infty$.

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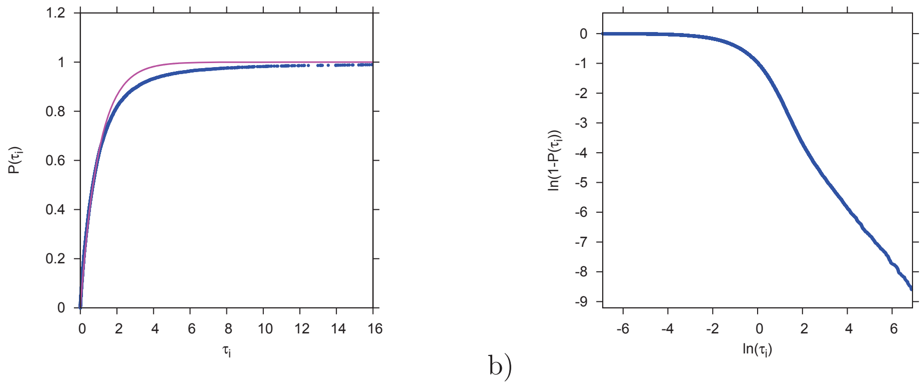

a) b)

Figure 2: Random numbers drawn for exponent $\nu = 0.9$. In a) ordered data in blue and as curve the cumulative distribution function for the exponential waiting time with $\nu = 1$. In b) we plot the same data as in a) but now as $ln(1 - P(\tau_i))$ over $ln(\tau_i)$.

The asymptotics for large arguments $\tau \to \infty$, see in Fig. 3 a) as red line giving the power law tail, as well as for small arguments $\tau \to 0$, see in Fig. 3 a) as green line with the functional form of a streched exponential, can be given analytically [2], as well as the complete functional form of Ψ in terms of the Mittag-Leffler function, see in Fig. 3 a) the interpolating curve between the two asymptotical curves. These analytic results compare well with the data plot as given in Fig. 2 b), as can be observed here in Fig. 3 b).

Finally we plot from the random number drawing the result also for the exponent $\nu = 0.5$, see Fig. 4, with now less pronounced shoulder for small τ-values and the power law for large τ appearing earlier.

3.2 Drawing random numbers from power law distributions: Jump distribution

The same as for the waiting times can, of course, also be performed for the jump distribution $\lambda(\xi)$ with a function $\Lambda(\xi)$ analogously to the survival function $\Psi(\tau)$, now with

$$\Lambda(\xi) := \int_{\xi}^{\infty} \lambda(\tilde{\xi})\, d\tilde{\xi} = 1 - \int_{-\infty}^{\xi} \lambda(\tilde{\xi})\, d\tilde{\xi} \tag{31}$$

hence the probability to have jumps larger than ξ. In order to be able to perform the double logarithmic plot we take the modulus of ξ, but otherwise the procedure described above holds here equally, see Fig. 5, for the exponent $\mu = 1.8$, hence close to the Gaussian case with $\mu = 2$.

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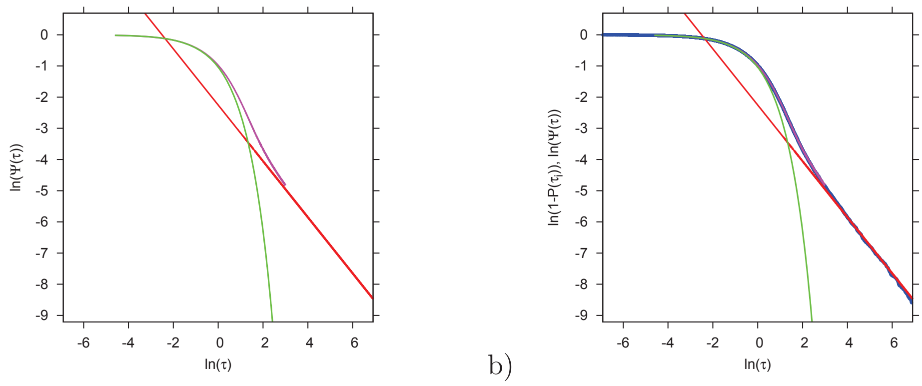

a) b)

Figure 3: The analytical expression as well as the asymptotics can be given explicitly, see a), and compare well with the plot from the drawn random numbers, see b).

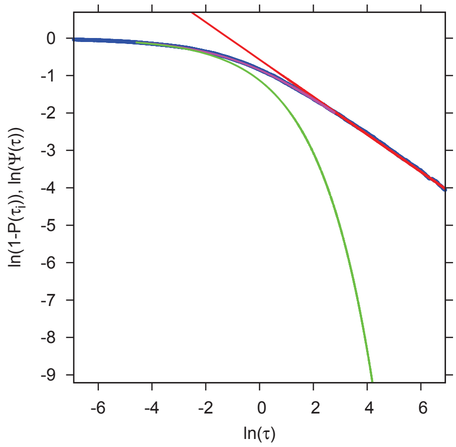

Figure 4: Results now for $\nu = 0.5$ in the same way as was given for $\nu = 0.9$ previously.

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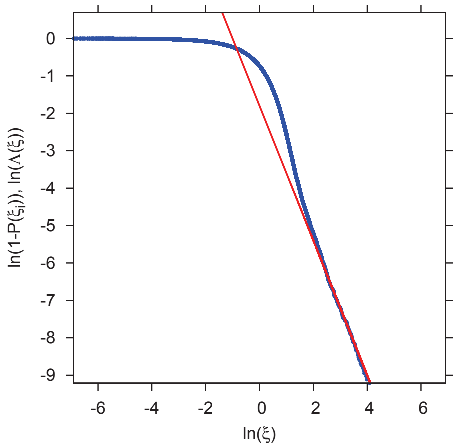

Figure 5: The equivalent $\Lambda(\xi)$ to the survivial function $\Psi(\tau)$ now applied to the spatial jump distribution $\lambda(\xi)$ in double logarithmic plot. A similar qualitative picture is observed as for $\Psi(\tau)$, and now the power law asymptotics is analogously $|\xi|^{-\mu}$, see red line.

3.3 Drawing random numbers from power law distributions: Random walk distribution

Finally, we investigate the random walk of combined jumps and waiting times in the same way as for the individual distributions, see Fig. 5. The probability $p(x,t)$ is plotted in its complementary distribution function for the time $t=2$ in double logarithmic scale.

With the here presented methods of generating power law distributions and analyzing them, e.g. via the complementary cumulative distribution function, now empirical data such as human mobility data can be readily investigated and compared easily with theoretical statements on scaling behaviour.

4 Power laws in human mobility to describe epidemiological spreading

Besides analysing the often very difficult to obtain direct human mobility data, there also have been numerous modelling approaches suggested in geography since the 1940s to use the easier to obtain population density information and model from this the related human mobility. A more recent approach is the so called radiation model [7], which agrees well with some human mobility data respectively surrogates for human mobility and human contacts, as needed as input for any spatially extended epidemiological modelling. Especially, power law behaviour is being reported from the radiation model, and compares well with empirical

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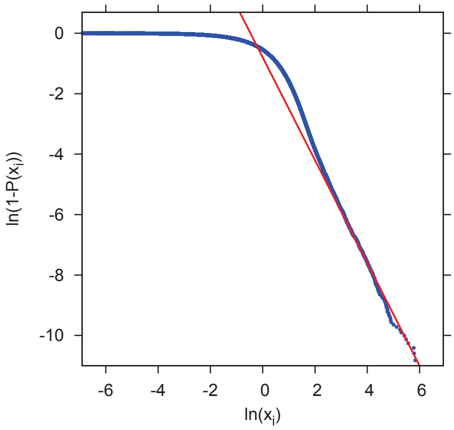

Figure 6: The complementary cumulative distribution function $(1-P(x_i,t))$ of the random walk in double logarithmic plot, and its asymptotics of a power law with exponent $\mu = 1.7$. The time is $t = 2$ with a waiting time exponent $\nu = 0.8$.

data [7]. Analogously to radiation emission and absorption in physics, also in the radiation model for human mobility the "emission and absorption" probabilities of people from location x_i to location x_j depend also on population densities inbetween the two locations and not only on the densities in the locations and their distance, like it was e.g. in the older gravitation model. Future work will be dedicated to investigate in how far the power laws predicted by models like the radiation model coincide with actual information on human mobility and in a second step in how far these informations help in the spatially extended modelling of e.g. dengue fever in Thailand, for which especially good and long term data sets are becoming available [9].

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